Periodic solutions for nonlinear telegraph equation via elliptic regularization

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Abstract. In this work we are concerned with the existence and uniqueness of $T$-periodic weak solutions for an initial-boundary value problem associated with nonlinear telegraph equations type in a domain $Q \subset \mathbb{R}^N$. Our arguments rely on elliptic regularization technics, tools from classical functional analysis as well as basic results from theory of monotone operators.

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1 Introduction and description of the elliptic regularization method

In this paper we deal with the existence of time-periodic solutions for the nonlinear telegraph equation

$$w'' + w' - \Delta w + w + |w'|^{p-2}w' = f, \quad (x, t) \in Q = \Omega \times ]0, T[. \quad (1.1)$$

$\Omega$ being a bounded domain in $\mathbb{R}^N$ with a sufficiently regular boundary $\partial \Omega$.

All derivatives are in the sense of distributions, and by $\xi'$ it denotes $\frac{\partial \xi}{\partial t}$. The function $f$ we will be assumed as regular as necessary.

We shall use, throughout this paper, the same terminology of the functional spaces used, for instance, in the books of Lions [6]. In particular, we denote by
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\[ V = H_0^1(\Omega) \text{ and } H = L^2(\Omega). \]  

The Hilbert space \( V \) has inner product \((.,.)\) and norm \(\|\cdot\|\) given by \( (u, v) = \int_\Omega \nabla u . \nabla v \, dx, \|u\|^2 = \int_\Omega |\nabla u|^2 \, dx. \) For the Hilbert space \( H \) we represent its inner product and norm, respectively, by \((.,.)\) and \(|.|\), defined by \( (u, v) = \int_\Omega uv \, dx, \|u\|^2 = \int_\Omega |u|^2 \, dx. \)

The telegraph equation appears when we look for a mathematical model for the electrical flow in a metallic cable. From the laws of electricity we deduce a system of partial differential equations where the unknown are the intensity of current \( i \) and the voltage \( u \), cf. Courant-Hilbert [4], p. 192–193, among others.

By algebraic calculations we eliminate \( i \) and we get the partial differential equation:

\[ \frac{\partial^2 u}{\partial t^2} - C^2 \frac{\partial^2 u}{\partial x^2} + (\alpha + \beta) \frac{\partial u}{\partial t} + \alpha \beta u = 0, \]

called Telegraph Equation. In this case the coefficients \( C, \alpha, \beta \) are constants.

Motivated by this model, Prodi [10] investigated the existence of periodic solution in \( t \) for the equations

\[ \frac{\partial^2 u}{\partial t^2} - \Delta u + u + \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} = f, \]

in a bounded open set \( \Omega \) of \( R^N \) with Dirichlet zero conditions on the boundary.


\begin{align*}
|w'' - \Delta w + \gamma(w')| &= f \text{ in } Q = \Omega \times ]0, T[, \\
w &= 0 \text{ on } \Sigma = \partial \Omega \times ]0, T[, \\
w(0) &= w(T), \quad w'(0) = w'(T) \text{ in } \Omega,
\end{align*}

with \( \gamma(w') = |w'|^{p-2} w' \).

Because of this important physical background, the existence of time-periodic solutions of the telegraph equations with boundary condition for space variable \( x \) has been studied by many authors, see [7, 8, 9, 11] and the references therein.

We consider the existence of the solutions \( w(x, t) \) of Eq. (1.1), which satisfy the time-periodic (or \( T \)-periodic) condition

\[ w(0) = w(x, T), \quad w'(x, 0) = w'(T), \quad x \in \Omega, \]

subject to the Dirichlet condition

\[ w(x, t) = 0, \quad (x, t) \in \partial \Omega \times ]0, T[. \]  

(1.4)

Based on physical considerations, we restrict our analysis to the two dimensional space and standard hypothesis on \( f \) is assumed. Arguments within this paper are inspired by the work by Lions [6].

However, the classical energy method approach cannot be employed straightly, giving raise to a new mathematical difficulty. In fact, multiplying both sides of the equation (1.1) by \( w' \) and integrating on \( Q \), we have, using the periodicity condition, that

\[
\int_Q |w'(x, t)|^2 dx dt + \int_Q |w'(x, t)|^p dx dt = \int_Q f(x, t)w'(x, t) dx dt.
\]

In this way we obtain only estimates for

\[
\int_Q |w'(x, t)|^2 dx dt \quad \text{and} \quad \int_Q |w'(x, t)|^p dx dt,
\]

which is not sufficient to obtain solution for (1.1).

In view of this, as in Lions [6], we use an approach due to Prodi [10] which relies heavily on the following set of ideas: we investigate solutions for (1.1) of the type

\[
\begin{align*}
|w| &= u + u_0, \\
u_0 \text{ independent of } t \\
\int_0^T u(t) dt &= 0, \quad \text{the average of } u \text{ is zero.}
\end{align*}
\]

(1.5)

Substituting \( w \) given by (1.5) in (1.1), we obtain

\[
u'' + u' - \Delta u + u + |u'|^{p-2} u' = f + \Delta u_0 - u_0,
\]

(1.6)

which contains a new unknown \( u_0 \), independent of \( t \) by definition.

To eliminate \( u_0 \) in (1.6) we consider the derivative of (1.6) with respect to \( t \) obtaining

\[
\begin{align*}
\frac{d}{dt} (u'' + u' - \Delta u + u + |u'|^{p-2} u') &= \frac{df}{dt} \\
\int_0^T u(t) dt &= 0 \\
|u(0) = u(T), \quad u'(0) = u'(T). \end{align*}
\]

(1.7)
Suppose that we have found $u$ by (1.7). Observe that by (1.7),
\[ \frac{d}{dt}(u'' + u' - \Delta u + u + |u'|^{p-2}u' - f) = 0. \]
Thus $u$ is solution of
\[ u'' + u' - \Delta u + u + |u'|^{p-2}u' - f = g_0, \quad (1.8) \]
g_0 independent of $t$, in which $g_0$ is a known function.
Then $u_0$ is obtained as the solution of the Dirichlet problem:
\[ \left| \begin{array}{l}
-\Delta u_0 + u_0 = -g_0 \\
u_0 = 0 \text{ on } \partial\Omega.
\end{array} \right. \quad (1.9) \]
Therefore, $w = u + u_0$ is the T – periodic solution of (1.1). We are going to resolve problem (1.7) by using elliptic regularization.
Observe that Lions [6] investigate the problem (1.2) by elliptic regularization, reducing the problem to the theory of monotonous operators, cf. Lions [6].
In this work we consider the time – periodic problem (1.1), (1.3) and (1.4) and solve it by elliptic regularization as an application of the monotony type results, cf. Browder [3]. Thus our proof is a simpler alternative to the earlier approaches existing in the current literature.
In fact, we consider the periodic problem
\[ \left| \begin{array}{l}
w'' + w' - \Delta w + w + |w'|^{p-2}w' = f \quad \text{in } Q = \Omega \times ]0, T[, \\
w = 0 \text{ on } \partial\Omega \times ]0, T[, \\
w(x, 0) = w(x, T), \quad w'(x, 0) = w'(x, T) \quad \text{in } \Omega.
\end{array} \right. \quad (1.10) \]
Thus for $w = u + u_0$, the function $u$ is determined by (1.7).
We begin the functional space
\[ W = \left\{ v; \quad v \in L^2(0, T; V), \quad v' \in L^2(0, T; V) \cap L^p(Q), \\
v'' \in L^2(0, T; H), \quad \int_0^T v(s)ds = 0, \quad v(0) = v(T), \quad v'(0) = v'(T) \right\}. \quad (1.11) \]
The Banach structure of $W$ is defined by
\[ \|v\|_W = \|v\|_{L^2(0, T; V)} + \|v'\|_{L^2(0, T; V)} + \|v'\|_{L^p(0, T; L^p(\Omega))} + \|v''\|_{L^2(0, T; H)}. \]
In the sequel by $\langle \cdot, \cdot \rangle$ we will represent the duality pairing between $X$ and $X'$, $X'$ being the topological dual of the space $X$, and by $c$ (sometimes $c_1, c_2, \ldots$) we denote various positive constants.

Motivated by (1.7) we define the bilinear form $b(u, v)$ for $u, v \in W$ by

$$b(u, v) = \int_0^T \left[ (u'' + u' + u, v') + (Au, v') + \langle \gamma(u'), v' \rangle \right] dt,$$

where $A = -\Delta$ and $\gamma(u') = |u'|^{p-2}u'$.

Then the weak formulation of (1.7) is to find $u \in W$ such that

$$b(u, v) = \int_0^T (f, v') dt,$$  \hspace{1cm} (1.12)

for all $v \in W$.

Let us point out that the main difficulty in applying standard techniques from classical functional is due to the fact that the bilinear form $b(u, v)$ is not coercive. To resolve this issue, we perform an elliptic regularization on $b(u, v)$, following the ideas of Lions [6]. Subsequently we apply Theorem 2.1, p. 171 of Lions [6] to finally establish existence and uniqueness of solution to elliptic problem (1.12).

2 Main result

As we said in the Section 1, the method developed in this article is a variant of the elliptic regularization method introduced in Lions [6] in the context of the telegraph equation.

Indeed, following the same type of reasoning cf. Lions [6], to obtain the elliptic regularization, given $\mu > 0$ and $u, v \in W$ we define

$$\pi_\mu(u, v) = \mu \int_0^T \left[ (u'', v'') + (u', v') + (Au', v') \right] dt + \int_0^T (u'' + u' + Au + u + \gamma(u'), v') dt,$$  \hspace{1cm} (2.1)

where $A = -\Delta$ and $\gamma(u') = |u'|^{p-2}u'$. 

It is easy to see, cf. Lemma 2.2, that the application $v \to \pi_{\mu}(u, v)$ is continuous on $W$. This allows to build a linear operator $B_{\mu} : W \to W', \langle B_{\mu}(u), v \rangle = \pi_{\mu}(u, v)$.

As we shall see, the linear operator $B_{\mu}$ satisfies the following properties:

(a) $B_{\mu}$ is a strictly monotonous operator; $\langle B_{\mu}(v) - B_{\mu}(z), v - z \rangle > 0$ for all $v, z \in W, v \neq z$;

(b) $B_{\mu}$ is a hemicontinuous operator; $\lambda \to \langle B_{\mu}(v + \lambda z), w \rangle$ is continuous in $\mathbb{R}$;

(c) $B_{\mu}(S)$ is bounded in $W'$ for all bounded set $S$ in $W$;

(d) $B_{\mu}$ is coercive; $\frac{\langle B_{\mu}(v), v \rangle}{|v|_W} \to \infty$ as $|v|_W \to \infty$.

In view of these properties and as consequence of Theorem 2.1, p. 171 of Lions [6], the existence and uniqueness of a function $u_{\mu} \in W$ such that

$$\pi_{\mu}(u_{\mu}, v) = \int_0^T (f, v') \, dt, \quad \text{for all } v \in W,$$

follows immediately.

The Eq. (2.2) is called of elliptic regularization of problem (1.7).

Our main result is as follows

**Theorem 2.1.** Suppose $f \in L^{p'}(0, T; L^{p'}(\Omega))$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $p > 2$. Then there exists only one real function $w = w(x, t), (x, t) \in Q, w \in W$, such that

$$w = u + u_0, \quad u_0 \in H_0^1(\Omega) \quad (2.3)$$

$$u \in L^2(0, T; V) \quad (2.4)$$

$$u' \in L^p(0, T; L^p(\Omega)) \quad (2.5)$$

and $w$ satisfying (1.1) in the sense of $L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega))$.

Now, we begin by stating some lemmas that will be used in the proof of the Theorem 2.1.
Lemma 2.1. If \( \int_0^T u(x, t) dt = 0 \) then
\[
\int_0^T \|u\|_V^2 dt \leq C \int_0^T \|u'\|_{L^p(\Omega)}^p dt \quad \text{and} \quad \int_0^T \|u\|_{L^2(0, T)}^2 dt \leq C \int_0^T \|u'\|_{L^p(\Omega)}^p dt,
\]
for \( u \) derivable with respect to \( t \) in \([0, T]\) and \( u \in L^2(0, T; V), u' \in L^2(0, T; V) \cap L^p(0, T; L^p(\Omega)) \).

Proof. The proof of Lemma 2.1 can be obtained with slight modifications from Lions [6] or Medeiros [8].

Lemma 2.2. The form \( v \rightarrow \pi_\mu(u, v) \) defined in (2.1) is continuous on \( W \).

Proof. By Cauchy-Schwarz inequality and Young’s inequality we have
\[
|\pi_\mu(u, v)| \leq c_\mu \|u\|_W \|v\|_{W'},
\]
where \( c_\mu \) is a constant positive that depend of \( \mu \). Then the result follows. \( \Box \)

Lemma 2.3. The operator \( B_\mu : W \rightarrow W', \langle B_\mu(u), v \rangle = \pi_\mu(u, v) \) is hemi-continuous, bounded, coercive and strictly monotonous from \( W \rightarrow W' \).

Proof. It follows of (2.6) that \( B_\mu(u) \) is bounded. From Lemma 2.1 and equality \( \int_0^T (\gamma(u'), u') dt = \|u'\|_{L^p(\Omega)}^p \), we obtain
\[
\langle B_\mu(v), v \rangle \geq c_\eta \|v\|_{W'}^2,
\]
because \( \int_0^T u(x, t) dt = 0 \). Thus \( B_\mu \) is \( W \)-coercive. The hemicontinuity of the operator \( v \rightarrow |v|^{p-2}v \) allow us to conclude that the operator \( B_\mu \) is hemicontinuous. Finally, the proof that the operator \( B_\mu \) is strictly monotonous follows as in Lions [6], p. 494. \( \Box \)

Proof of Theorem 2.1. The arguments above show that there exists a unique solution \( u_\mu \in W \) of the elliptic problem (2.2).

Explicitly the Eq. (2.2) has the form:

\[
\mu \int_0^T \left[ (u''_\mu, \nu') + (u'_\mu, \nu') + ((u'_\mu, \nu')) \right] dt + \int_0^T (u''_\mu + u'_\mu + u_\mu, \nu') dt \\
+ \int_0^T (u_\mu, \nu') dt + \int_0^T \langle \gamma(u'_\mu), \nu' \rangle dt = \int_0^T (f, \nu') dt.
\]

(2.7)

We need let \( \mu \) goes to zero in order to obtain \( u_\mu \rightharpoonup u \) for the solution. Then we need estimates for \( u_\mu \).

In fact, setting \( v = u_\mu \) in (2.7) and observing that \( u_\mu \) and \( u'_\mu \) are periodic since they belongs to \( W \), we obtain

\[
\mu \int_0^T \left( |u''_\mu|^2 + |u'_\mu|^2 + \|u_\mu\|^2 \right) dt + \int_0^T |u'_\mu|^2 dt + \int_0^T \|u'_\mu\|^p_{L^p(\Omega)} dt \\
\leq \frac{1}{\epsilon p'} \int_0^T \|f\|_{L^{p'}(\Omega)}^{p'} dt + \frac{\epsilon}{p} \int_0^T |u'_\mu|^p_{L^p(\Omega)} dt.
\]

(2.8)

This implies that

\[
(u'_\mu) \text{ is bounded in } L^2(0, T; H) \text{ when } \mu \to 0
\]

(2.9)

\[
(u'_\mu) \text{ is bounded in } L^p(0, T; L^p(\Omega)) \text{ when } \mu \to 0
\]

(2.10)

\[
\mu \int_0^T \left( |u''_\mu|^2 + |u'_\mu|^2 + \|u'_\mu\|^2 \right) dt \leq c_1
\]

(2.11)

Since \( \int_0^T u_\mu dt = 0 \), we have by Lemma 2.1 that

\[
(u_\mu) \text{ is bounded in } L^p(0, T; L^p(\Omega))
\]

(2.12)

\[
\mu \int_0^T \|u_\mu\|^2 dt \leq c_2.
\]

(2.13)

Setting

\[
v(t) = \int_0^t u_\mu(\sigma) d\sigma - \frac{1}{T} \int_0^T (T - \sigma)u_\mu(\sigma) d\sigma,
\]

(2.14)
it implies
\[
\int_0^T v(t) \, dt = 0, \quad \forall v \in W
\tag{2.15}
\]
In fact, integrating both sides of the equation (2.14) on \([0, T]\), we obtain
\[
\int_0^T v(t) \, dt = \int_0^T \int_0^T u(\sigma) \, d\sigma \, dt - \int_0^T \int_0^T (T - \sigma)u(\sigma) \, d\sigma \, dt.
\]
On the other hand,
\[
\int_0^T \int_0^T (T - \sigma)u(\sigma) \, d\sigma \, dt = \int_0^T \int_0^T (T - \sigma)u(\sigma) \, d\sigma
\]
\[
= (T - \sigma) \int_0^T u(s) \, ds \bigg|_0^T + \int_0^T \int_0^\sigma u(s) \, ds \, d\sigma = \int_0^T \int_0^\sigma u(s) \, ds \, d\sigma.
\]
Therefore, we reach our aim (2.15).

Thus, taking into account (2.14) in (2.2) we get
\[
\mu \int_0^T [(u'' \mu, u'' \mu) + (u' \mu, u' \mu) + (Au' \mu, u' \mu)] \, dt
\]
\[
+ \int_0^T [(u'' \mu, u' \mu) + (u' \mu, u' \mu) + (Au' \mu, u' \mu) + (u' \mu, u' \mu) + (\gamma(u' \mu), u' \mu)] \, dt
\tag{2.16}
\]
\[
= \int_0^T (f, u \mu) \, dt.
\]

By using periodicity of \(u \mu, u'' \mu \in W\), we obtain
\[
\int_0^T (u'' \mu, u' \mu) \, dt = \int_0^T (u' \mu, u \mu) \, dt = \int_0^T (Au' \mu, u \mu) \, dt = 0.
\tag{2.17}
\]
On the other hand,
\[
\int_0^T (u'' \mu, u' \mu) \, dt = (u'' \mu(T), u' \mu(T)) - (u'' \mu(0), u' \mu(0))
\]
\[
- \int_0^T (u' \mu, u' \mu) \, dt = - \int_0^T |u' \mu|^2 \, dt.
\tag{2.18}
\]
From (2.17), (2.18) and estimate (2.9), we have
\[
\left| \int_0^T (u'' \mu, u' \mu) \, dt \right| \leq c_2 \quad \text{when } \mu \to 0. \tag{2.19}
\]
Also, from (2.10) and (2.12) we obtain
\[
\int_0^T |u'_\mu|^2 \, dt + \int_0^T (\gamma(u'_\mu), u_\mu) \, dt 
\leq \int_0^T |u'_\mu|^2 \, dt + \|\gamma(u'_\mu)\|_{L^{p'}(0,T;L^{p'}(\Omega))} \|u_\mu\|_{L^p(0,T;L^p(\Omega))} \leq c_3.
\] (2.20)

Combining (2.17), (2.19) and (2.20) with (2.16) we deduce
\[
\int_0^T \|u'_\mu\|^2 \, dt \leq c_4.
\] (2.21)

It follows from (2.21) and (2.10) that there exists a subsequence from \((u_\mu)\), still denoted by \((u_\mu)\), such that
\[
\begin{align*}
  u_\mu &\longrightarrow u \text{ weak in } L^2(0, T; V) \\
  u'_\mu &\longrightarrow u' \text{ weak in } L^p(0, T; L^p(\Omega)) \\
  \gamma(u'_\mu) &\longrightarrow \chi \text{ weak in } L^{p'}(0, T; L^{p'}(\Omega)).
\end{align*}
\] (2.22) (2.23) (2.24)

Our next goal is to show that \(u\) verifies (1.7)_2 \(\sim\) (1.7)_3.

Indeed, it follows from (2.22) and (2.23) that \(u_\mu \in C^0([0, T]; H)\) and
\[
\lim_{\mu \to 0} \int_0^T (u'_\mu, \varphi) \, dt = \int_0^T (u', \varphi) \, dt, \ \forall \varphi \in L^2(0, T; H)
\] (2.25)
\[
\lim_{\mu \to 0} \int_0^T (u_\mu, \varphi) \, dt = \int_0^T (u, \varphi) \, dt, \ \forall \varphi \in L^2(0, T; V)
\] (2.26)

Setting \(\varphi = \theta v\) into (2.25) with \(\theta \in C^1([0, T]; \mathbb{R})\), \(\theta(0) = \theta(T)\) and \(v \in V\), we have
\[
\int_0^T (u'_\mu, \theta v) \, dt \longrightarrow \int_0^T (u', \theta v) \, dt
\] (2.27)
\[
\int_0^T (u_\mu, \theta' v) \, dt \longrightarrow \int_0^T (u, \theta' v) \, dt.
\] (2.28)

Again, by using periodicity of \(u_\mu\) and \(u'_\mu\), we obtain
\[
\int_0^T \frac{d}{dt} (u_\mu, \theta v) \, dt = (u_\mu(T), \theta(T)v) - (u_\mu(0), \theta(0)v) = 0.
\]
Thus
\[ \int_0^T (u'_\mu, \theta v) \, dt + \int_0^T (u_\mu, \theta' v) \, dt = 0. \]

Since
\[ \int_0^T (u', \theta v) \, dt + \int_0^T (u, \theta' v) \, dt = 0, \]
as \( \mu \to 0 \), we obtain
\[ \int_0^T \frac{d}{dt}(u, \theta v) \, dt = 0. \]

This implies that
\[ (u(T), \theta(T)v) - (u(0), \theta(0)v) = 0, \]
that is,
\[ u(T) = u(0). \] (2.29)

The proof that \( u'(0) = u'(T) \) will be given later. Now, we go to prove that
\[ \int_0^T u(t) \, dt = 0. \]

Taking the scalar product on \( H \) of \( \int_0^T u_\mu(\sigma) \, d\sigma = 0 \) with \( \varphi(t), \varphi \in L^2(0, T; H) \), we find
\[ \left( \int_0^T u_\mu(\sigma) \, d\sigma, \varphi(t) \right) = 0. \]

Thus
\[ \int_0^T (u_\mu(\sigma), \varphi(t)) \, d\sigma = 0. \]

Therefore,
\[ \int_0^T (u(\sigma), \varphi(t)) \, d\sigma = \left( \int_0^T u(\sigma) \, d\sigma, \varphi(t) \right) = 0, \quad \forall \varphi(t) \in H, \] (2.30)
as \( \mu \to 0 \).

It follows from (2.30) that
\[ \int_0^T u(t) \, dt = 0. \] (2.31)
From (2.9), (2.10), (2.11) and (2.13), we deduce

\[ u'_\mu \rightharpoonup u' \text{ weak in } L^2(0, T; H), \]  
\[ u''_\mu \rightharpoonup u'' \text{ weak in } L^p(0, T; L^p(\Omega)), \]  
\[ \sqrt{\mu}u''_\mu \rightharpoonup \chi_1 \text{ weak in } L^2(0, T; H), \]  
\[ \sqrt{\mu}u'u_\mu \rightharpoonup \chi_2 \text{ weak in } L^2(0, T; H), \]  
\[ \sqrt{\mu}u'u_\mu \rightharpoonup \chi_3 \text{ weak in } L^2(0, T; V). \]

(2.32) \hspace{1cm} (2.33) \hspace{1cm} (2.34) \hspace{1cm} (2.35) \hspace{1cm} (2.36)

It follows from (2.34) that

\[ \lim_{\mu \to 0} \sqrt{\mu} \int_0^T (u''_\mu, \varphi) \, dt = \int_0^T (\chi_1, \varphi) \, dt \quad \forall \varphi \in L^2(0, T; H). \]  

(2.37)

Hence, taking \( \varphi = v'' \), \( v \in W \), in (2.37), we find

\[ \lim_{\mu \to 0} \sqrt{\mu} \int_0^T (u''_\mu, v'') \, dt = \int_0^T (\chi_1, v'') \, dt. \]

Therefore

\[ \lim_{\mu \to 0} \mu \int_0^T (u''_\mu, v'') \, dt = \lim_{\mu \to 0} \sqrt{\mu} \left( \int_0^T (u''_\mu, v'') \, dt \right) = 0. \]  

(2.38)

By analogy, we prove that

\[ \lim_{\mu \to 0} \mu \int_0^T (u'_\mu, v') \, dt = \lim_{\mu \to 0} \mu \int_0^T (Au'_\mu, v') \, dt = 0. \]  

(2.39)

By using periodicity of \( u_\mu, v \in W \), we obtain

\[ \int_0^T \frac{d}{dt}(u'_\mu, v') \, dt = 0. \]

This implies that

\[ \int_0^T (u''_\mu, v') \, dt = - \int_0^T (u'_\mu, v'') \, dt. \]  

(2.40)

It follows of (2.9) that

\[ \int_0^T (u'_\mu, \varphi) \, dt \rightharpoonup \int_0^T (u', \varphi) \, dt \quad \forall \varphi \in L^2(0, T; H). \]  

(2.41)
Taking \( \varphi = v'' \in L^2(0, T; H) \) in (2.41) we obtain

\[
\lim_{\mu \to 0} \int_0^T (u'_\mu, v'') \, dt = \int_0^T (u', v'') \, dt.
\]  \hfill (2.42)

From (2.2), we can write

\[
\mu \int_0^T [(u''_\mu, v'') + (u'_\mu, v') + (Au'_\mu, v')] \, dt
\]

\[
+ \int_0^T [(u''_\mu, v') + (u'_\mu, v') + (Au'_\mu, v') + (u'_\mu, v') + (\gamma(u'_\mu), v')] \, dt
\]

\[
= \int_0^T (f, v') \, dt.
\]  \hfill (2.43)

From (2.9), (2.10), (2.22), (2.38), (2.39), (2.40) and (2.42), we can pass to the limit in (2.43) when \( \mu \to 0 \) and obtain

\[
\int_0^T \left[ (u', v'') + (u', v') + (Au', v') + (u, v') + (\chi, v') \right] \, dt
\]

\[
= \int_0^T (f, v') \, dt, \quad \forall v \in W.
\]  \hfill (2.44)

Let \( (\rho_v) \) be a regularizing sequence of even periodic functions in \( t \), with period \( T \).

Denote by \( \tilde{v} = u \ast \rho_v \ast \rho_v \), where \( \ast \) is the convolution operator. Integrating by parts, we find \( u' \ast \rho_v \ast \rho_v = u \ast \rho'_v \ast \rho_v \).

Observe by (2.12) and (2.21) that \( \tilde{v} \in C^\infty(\mathbb{R}; V) \), \( \tilde{v}' \in C^\infty(\mathbb{R}; L^p(\Omega)) \), \( \tilde{v}'' \in C^\infty(\mathbb{R}; H) \), \( v \) and \( \tilde{v}'' \) periodic in \( t \).

As in Brézis [2], p. 67, we to show that

\[
\int_0^T (u', \tilde{v}'') \, dt = 0.
\]  \hfill (2.45)

In fact, we have

\[
\int_0^T \frac{d}{dt}(u', u' \ast \rho_v \ast \rho_v) \, dt = \int_0^T (u'', u' \ast \rho_v \ast \rho_v) + \int_0^T (u', u'' \ast \rho_v \ast \rho_v) \, dt
\]

\[
= 2 \int_0^T (u', u' \ast \rho'_v \ast \rho_v) \, dt = 2 \int_0^T (u', \tilde{v}'') \, dt.
\]
As
\[\int_0^T (u', u' \ast \rho_v) \, dt = \int_0^T \frac{1}{2} \frac{d}{dt} (u', u' \ast \rho_v) \, dt = 0,\]
due to periodicity of \(u'\) and \(\rho_v\), it follows (2.45).

Similarly, we show that
\[\int_0^T (u', \tilde{v}') \, dt = 0.\] (2.46)
\[\int_0^T (Au, \tilde{v}') \, dt = 0.\] (2.47)
\[\int_0^T (u, \tilde{v}') \, dt = 0.\] (2.48)

From (2.44) to (2.48) we obtain
\[\int_0^T (\chi, u') \, dt = \int_0^T (f, u') \, dt.\] (2.49)

Now, let us prove that \(\chi = \gamma(u')\).

In fact, from (2.2) and (2.1) we get
\[
\mu \int_0^T [\|u''_\mu\|^2 + |u'_\mu|^2 + \|u'_\mu\|^2] \, dt + \int_0^T [\|u'_\mu\|^2 + (\gamma(u'_\mu), u'_\mu)] \, dt
= \int_0^T (f, u'_\mu) \, dt.\]

(2.50)

We define
\[X_\mu = \int_0^T (\gamma(u'_\mu) - \gamma(\varphi), u'_\mu - \varphi) \, dt\]
\[+ \mu \int_0^T [\|u''_\mu\|^2 + |u'_\mu|^2 + \|u'_\mu\|^2] \, dt\]
\[+ \int_0^T |u'_\mu|^2 \, dt, \, \forall \varphi \in L^p(0, T; L^p(\Omega))\]

(2.51)

It follows from (2.50) and (2.51) that
\[X_\mu = \int_0^T (f, u'_\mu) \, dt - \int_0^T (\gamma(\varphi), u'_\mu - \varphi) \, dt - \int_0^T (\gamma(u'_\mu), \varphi) \, dt.\]

(2.52)
From the convergences above, we get

\[ X_\mu \rightarrow X = \int_0^T (f, u') dt - \int_0^T (\gamma(\varphi), u' - \varphi) dt - \int_0^T (\chi, \varphi) dt. \quad (2.53) \]

Taking into account (2.53) into (2.49) yields

\[ X = \int_0^T (\chi, u') dt - \int_0^T (\gamma(\varphi), u' - \varphi) dt - \int_0^T (\chi, \varphi) dt. \quad (2.54) \]

Combining (2.53) and (2.54), we obtain

\[ X = \int_0^T (\chi - \gamma(\varphi), u' - \varphi) dt. \quad (2.55) \]

Since \( X_\mu \geq 0 \), for all \( \varphi \in L^p(0, T; L^p(\Omega)) \), then \( X \geq 0 \).

Thus,

\[ \int_0^T (\chi - \gamma(\varphi), u' - \varphi) dt \geq 0, \quad \forall \varphi \in L^p(0, T; L^p(\Omega)). \quad (2.56) \]

Since \( \gamma : L^p(0, T; L^p(\Omega)) \rightarrow L^{p'}(0, T; L^{p'}(\Omega)), \gamma(u') = |u'|^{p-2}u' \), is hemicontinuous operator, the inequality above implies \( \chi = \gamma(u') \). It is sufficient to set \( \varphi(t) = u'(t) - \lambda w(t), \lambda > 0, w \in L^p(0, T; L^p(\Omega)) \) arbitrarily and let \( \lambda \rightarrow 0 \).

We consider \( \psi \in C^\infty([0, T]; V \cap L^p(\Omega)) \) satisfying

\[ \begin{align*}
\int_0^T \psi \, dt &= 0, \\
\psi(0) &= \psi(T).
\end{align*} \quad (2.57) \]

Setting

\[ v(t) = \int_0^T \psi \, d\sigma - \frac{1}{T} \int_0^T (T - \sigma) \psi(\sigma) \, d\sigma \quad (2.58) \]

in (2.44), yields

\[ \int_0^T \left[ (-u', \psi) + (u', \psi) + (Au, \psi) + (u, \psi) + (\gamma(u'), \psi) - (f, \psi) \right] dt = 0, \quad (2.59) \]

because \( v'(t) = \psi(t), \quad v''(t) = \psi'(t) \).
In particular, choosing \( \psi = \theta' v \), with \( \theta \in D[0, T[ \) and \( v \in V \cap L^p(\Omega) \), in (2.59) we get

\[
\int_0^T \left[ (-u', \theta'' v) + (u', \theta' v) + (Au, \theta' v) + (u, \theta' v) + (\gamma(u'), \theta' v) - (f, \theta' v) \right] dt = 0, \quad \forall \theta \in D[0, T[, \ v \in V \cap L^p(\Omega),
\]

or equivalently,

\[
\int_0^T (u'' + u' + Au + u + \gamma(u') - f, v) \theta' dt = 0,
\]

for all \( v \in V \cap L^4(\Omega) \) and \( \theta \in D[0, T[ \).

Hence,

\[
\frac{d}{dt} [(u'' + u' + Au + u + \gamma(u') - f, v)] = 0, \quad \forall v \in V \cap L^p(\Omega).
\]

Consequently, there exists a function \( g_0 \) independent of \( t \) such that

\[
u'' + u' + Au + u + \gamma(u') - f = g_0, \text{ independent of } t.
\]

We verify that

\[
u''(\varphi) = \int_0^T u''(t)\varphi(t) dt = -\int_0^T u'(t)\varphi'(t) dt \in L^p(\Omega)
\]

\[
Au(\varphi) = \int_0^T (Au(t))\varphi(t) dt \in V'
\]

\[
\gamma(u')(\varphi) = \int_0^T \gamma(u')\varphi dt \in L^{p'}(\Omega)
\]

\[
u'(\varphi) = \int_0^T u'(t)\varphi(t) dt \in L^p(\Omega)
\]

\[
u(\varphi) = \int_0^T u(t)\varphi(t) dt \in L^2(\Omega)
\]

\[
f(\varphi) = \int_0^T f(t)\varphi(t) dt \in L^{p'}(\Omega),
\]

for all \( \varphi \in D[0, T[ \), because \( u' \in L^p(0, T; L^p(\Omega)) \).

Thus, from (2.63) to (2.68) and (2.62), we can write

\[
g_0 \int_0^T \varphi(t) \, dt \in V' + L^p(\Omega).
\]

Therefore

\[
g_0 \in V' + L^p(\Omega). \tag{2.69}
\]

It follows from (2.62) that

\[
u'' = f + g_0 - u' - Au - u - \gamma(u') \in L^2(0, T; V') + L^p(0, T; L^p(\Omega)). \tag{2.70}
\]

Hence, we deduce from (2.62) that,

\[
\int_0^T (u'' + u' + Au + u + \gamma(u') - f - g_0, \psi) \, dt = 0, \tag{2.71}
\]

with \( \psi \) given in (2.57).

Thus

\[
\int_0^T (u'' + u' + Au + u + \gamma(u') - f - g_0, \psi) \, dt
\]

\[
= \int_0^T \frac{d}{dt} (u'(t), \psi) \, dt + \int_0^T \left[ (-u'(t), \psi') + (u'(t), \psi) + (Au(t), \psi) + (u, \psi) + (\gamma(u'), \psi) - (f, \psi) - (g_0, \psi) \right] \, dt
\]

\[
= (u'(T), \psi(T)) - (u'(0), \psi(0)). \tag{2.72}
\]

Substituting (2.72) into (2.57) we obtain

\[
u'(0) = u'(T). \tag{2.73}
\]

Note that \( u'(0) \) and \( u'(T) \) make sense because \( u' \) and \( u'' \) belongs to \( L^p(0, T; L^p(\Omega)) \) and \( L^2(0, T; V') + L^p(0, T; L^p(\Omega)) \), respectively.

Let \( u_0 \) be defined by

\[
\begin{align*}
-\Delta u_0 + u_0 &= -g_0, \\
u_0 &= 0 \text{ on } \partial\Omega.
\end{align*} \tag{2.74}
\]
We recall that because \( n \leq 2 \) and \( p > 2 \), we have

\[
H_0^1(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^p'(\Omega) \hookrightarrow H^{-1}(\Omega) = V',
\]

where each space is dense in the following one and the injections are continuous.

This and (2.69) implies that \( g_0 \in H^{-1}(\Omega) = V' \).

Finally, we apply the Lax-Milgram Theorem to find a unique solution \( u_0 \in H_0^1(\Omega) \) of the Dirichlet problem (2.74).

Thus, \( w = u + u_0 \in L^2(0, T; V) \) with \( w' \in L^p(0, T; L^p(\Omega)) \) satisfies

\[
\begin{align*}
|w'' + w' - \Delta w + w + |w'|^{p-2}w' &= f \\
in L^2(0, T; V') + L^p(0, T; L^p(\Omega)), \\
w(0) = w(T) \\
w'(0) = w'(T),
\end{align*}
\]

that is, \( w \) is a T-periodic weak solutions of problem (1.1).

**Uniqueness.** Let us consider \( w_1 \) and \( w_2 \) be two functions satisfying Theorem 2.1 and let \( \xi = w_1 - w_2 \).

We subtract the equations (1.1) corresponding to \( w_1 \) and \( w_2 \) and we obtain

\[
\xi'' + \xi' + A\xi + \xi + \gamma(w_1') - \gamma(w_2') = 0. \tag{2.75}
\]

Denoting by \((\rho_\mu)\) the regularizing sequence defined above, by a similar argument used in the proof of existence of solutions for Theorem 2.1 we obtain

\[
\xi' * \rho_\mu * \rho_\mu = \xi' * \rho_\mu' * \rho_\mu. \tag{2.76}
\]

Hence, by using (2.3) and (2.4), we can write

\[
\xi = \psi + \xi_0, \quad \text{with} \quad \xi_0 \in V \quad \text{and} \quad \psi \in L^2(0, T; V). \tag{2.77}
\]

Also, from (2.76) we get

\[
\xi' * \rho_\mu * \rho_\mu = \xi' * \rho_\mu' * \rho_\mu = \psi' * \rho_\mu * \rho_\mu. \tag{2.78}
\]

Thus, we have by (2.5) that \( \psi' \in L^p(0, T; L^p(\Omega)) \). Therefore \( \xi' * \rho_\mu * \rho_\mu \) is periodic and

\[
\xi' * \rho_\mu * \rho_\mu \in C^\infty([0, T]; L^p(\Omega)). \tag{2.79}
\]
Then by (2.70) we can write

$$\xi'' \in L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega)).$$

This and (2.79) show that \(\int_0^T (\xi'', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt\) make sense and

$$\int_0^T (\xi'', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = 0. \quad (2.80)$$

Indeed,

$$\int_0^T \frac{d}{dt}(\xi', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = \int_0^T (\xi''', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt$$

$$+ \int_0^T (\xi', \xi'' \ast \rho_\mu \ast \rho_\mu) \, dt = \int_0^T (\xi''', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt$$

$$+ \int_0^T (\xi'', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt. \quad (2.81)$$

Therefore,

$$\int_0^T (\xi'', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = \frac{1}{2} \int_0^T \frac{d}{dt}(\xi', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = 0, \quad (2.82)$$

because \(\xi'\) and \(\rho_\mu\) are periodic.

Similarly

$$\int_0^T (A\xi, \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = 0 \quad (2.83)$$

$$\int_0^T (\xi', \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = 0 \quad (2.84)$$

$$\int_0^T (\xi, \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = 0. \quad (2.85)$$

Consequently, it follows from (2.75), (2.82), (2.83), (2.84) and (2.85) that

$$\int_0^T (\gamma(w'_1) - \gamma(w'_2), \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = 0. \quad (2.86)$$

Hence using (2.86), letting \(\mu\) tend to zero, we have

$$\int_0^T (\gamma(w'_1) - \gamma(w'_2), w'_1 - w'_2) \, dt = 0, \quad (2.87)$$

that is, \( w_1' = w_2' \).

This implies that
\[
\xi = w_1 - w_2 = \theta, \quad \theta \text{ independent of } t.
\]

Integrating the last equality on \([0, T]\) and observing that \( w_i = u_i + u_0 \) yields
\[
\int_0^T (w_1 - w_2) \, dt = \theta \int_0^T \, dt = \theta T = T(u_0_1 - u_0_2),
\]
because \( \int_0^T u_i \, dt = 0 \). Thus \( \theta \in V \).

It follows from (2.83) that
\[
\int_0^T (A\xi, \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = \int_0^T (A(w_1 - w_2), \xi' \ast \rho_\mu \ast \rho_\mu) \, dt = \int_0^T (A\theta, \theta' \ast \rho_\mu \ast \rho_\mu) \, dt = 0.
\]

This implies that, when \( \mu \to 0 \)
\[
\int_0^T (A\theta, \theta) = 0, \quad \forall \theta \in V.
\]

Therefore
\[
A\theta = 0, \quad \forall \theta \in V. \tag{2.88}
\]

Employing Green’s Theorem, we find
\[
(A\theta, \theta) = \int_\Omega -\Delta \theta \, dx = \int_\Omega (\nabla \theta)^2 \, dx - \int_{\Gamma} \theta \frac{\partial \theta}{\partial v} \, d\Gamma = \|\theta\|^2. \tag{2.89}
\]

Taking into account (2.89) into (2.88) yields \( \theta = 0 \), which proves the uniqueness of solutions of problem (1.2). Thus, the proof of Theorem 2.1 is complete. \( \square \)

**REFERENCES**


